

ON THE STRATIFICATION OF NONCOMMUTATIVE PRIME SPECTRA

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ABSTRACT. We study rational actions of an algebraic torus G by automorphisms on an associative algebra R . The G -action on R induces a stratification of the prime spectrum $\text{Spec } R$ which was introduced by Goodearl and Letzter. For a noetherian algebra R , Goodearl and Letzter showed that the strata of $\text{Spec } R$ are isomorphic to the spectra of certain commutative Laurent polynomial algebras. The purpose of this note is to give a new proof of this result which works for arbitrary algebras R .

INTRODUCTION

Let G be an affine algebraic group and let R be an associative algebra on which G acts rationally by algebra automorphisms. The induced G -action on the set $\text{Spec } R$ of all prime ideals of R leads to a stratification of $\text{Spec } R$ which was pioneered by Goodearl and Letzter [3]. For the special case of an algebraic torus G and a noetherian algebra R , Goodearl and Letzter have given a description of the strata of $\text{Spec } R$ in terms of the spectra of certain (commutative) Laurent polynomial algebras. Later, a different description of the strata was given in [6], for any algebra R and any connected affine algebraic group G . The purpose of this short note is to consolidate the description of [6], for the case of an algebraic torus G , with the earlier one due to Goodearl and Letzter, still working with a general algebra R .

Throughout, \mathbb{k} will be an algebraically closed base field of arbitrary characteristic. The largest G -stable ideal of the algebra R that is contained in a given ideal I of R , called the G -core of I , will be denoted by $I:G$; so $I:G = \bigcap_{g \in G} g.I$. It is easy to see [5, Proposition 8(b)] that, for any rational action of an affine algebraic group G on R , the collection of all G -cores of prime ideals of R coincides with the set of all G -prime ideals of R ; the latter set will be denoted by $G\text{-Spec } R$. If the algebraic group G is connected, then $G\text{-Spec } R$ is simply the set of all G -stable prime ideals of R [5, Proposition 19(a)]. The *Goodearl-Letzter stratification* of $\text{Spec } R$ is the partition

$$\text{Spec } R = \bigsqcup_{I \in G\text{-Spec } R} \text{Spec}_I R \quad \text{with} \quad \text{Spec}_I R = \{P \in \text{Spec } R \mid P:G = I\}.$$

For an algebraic torus G , we will give a description of each stratum $\text{Spec}_I R$ in terms of the spectrum of a suitable affine commutative algebra Z_I . The construction of the algebra Z_I and the precise statement of the main result will be given in Section 1 while the proof will occupy Section 2.

1. THE STRATIFICATION THEOREM

Let G be a connected affine algebraic group over \mathbb{k} and let R be an associative \mathbb{k} -algebra with a rational G -action by \mathbb{k} -algebra automorphism. For a given $I \in G\text{-Spec } R$, let $\mathcal{C}(R/I)$ denote the extended centroid of the algebra R/I ; this is a \mathbb{k} -field, called the *heart of I* , on which G acts via its action on R/I [5, 2.3]. We put

$$Z_I = \{c \in \mathcal{C}(R/I) \mid \text{the orbit } G.c \text{ spans a finite-dimensional } \mathbb{k}\text{-subspace of } \mathcal{C}(R/I)\}.$$

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Clearly, Z_I is a G -stable \mathbb{k} -subalgebra of $\mathcal{C}(R/I)$ and Z_I contains the subfield of G -invariants, $\mathcal{C}(R/I)^G$. Moreover, the G -action on Z_I is rational by [5, Lemma 18(b)].

We now focus on the case of an algebraic torus G . As usual, $X(G)$ will denote the lattice of rational characters of G . The following theorem, under the additional assumption that the algebra R is noetherian, is originally due to Goodearl and Letzter [3]; see also [2], [4] and [1, II.2.13].

Stratification Theorem. *Let G be an algebraic torus over \mathbb{k} that acts rationally by algebra automorphisms on the \mathbb{k} -algebra R , and let $I \in G\text{-Spec } R$. Then:*

(a) *There is an isomorphism*

$$Z_I \cong \mathcal{C}(R/I)^G \Gamma_I ,$$

the group algebra over the field $\mathcal{C}(R/I)^G$ of the sublattice $\Gamma_I = X(G/\text{Ker}_G(Z_I)) \subseteq X(G)$.

(b) *There is a G -equivariant order isomorphism*

$$\gamma: \text{Spec}_I R \xrightarrow{\sim} \text{Spec}(Z_I) .$$

2. PROOF OF THE STRATIFICATION THEOREM

2.1. Reductions and preliminaries. We begin with some remarks that hold for an arbitrary connected affine algebraic group G over \mathbb{k} . In order to describe the G -stratum $\text{Spec}_I R$ for a given $I \in G\text{-Spec } R$, we may replace R by R/I and thus assume that $I = 0$. In particular, R is a prime ring. We will write $\mathcal{C} = \mathcal{C}(R)$ and $Z = Z_0$ for brevity; so \mathcal{C} is a commutative \mathbb{k} -field on which G acts by automorphisms and

$$Z = \{c \in \mathcal{C} \mid \text{the orbit } G.c \text{ spans a finite-dimensional } \mathbb{k}\text{-subspace of } \mathcal{C}\} .$$

Our goal is to give a description of Z and to establish a suitable order isomorphism

$$\gamma: \text{Spec}_0 R = \{P \in \text{Spec } R \mid P:G = 0\} \xrightarrow{\sim} \text{Spec}(Z) .$$

We will need to consider various G -actions; they will usually be indicated by a simple dot as in the foregoing. When more precision is necessary, the G -action on \mathcal{C} will be denoted by ρ . The group G also acts on $\mathbb{k}[G]$, the algebra of regular functions of G , via the right and left regular representations $\rho_r, \rho_\ell: G \rightarrow \text{Aut}_{\mathbb{k}\text{-alg}}(\mathbb{k}[G])$; they are defined by $(\rho_r(x)f)(y) = f(yx)$ and $(\rho_\ell(x)f)(y) = f(x^{-1}y)$ for $x, y \in G$ and $f \in \mathbb{k}[G]$.

2.2. The algebra Z . Consider the Hopf \mathcal{C} -algebra

$$S = \mathcal{C} \otimes_{\mathbb{k}} \mathbb{k}[G] ;$$

this is an algebra of \mathcal{C} -valued functions on G via $(\sum_i c_i \otimes f_i)(g) = \sum_i c_i f_i(g)$. The group G acts on the ring S via $\rho \otimes \rho_r$. If $g \in G$ and $s = \sum_i c_i \otimes f_i \in S$ then $g.s \in S$ is the function $G \rightarrow \mathcal{C}$ that is given by

$$(g.s)(x) = \sum_i (g.c_i) f_i(xg) = g. \sum_i c_i f_i(xg) \quad (x \in G) .$$

Let S^G denote the subring of G -invariants in S . Thus, $s \in S^G$ if and only if $g^{-1}.s(1) = s(g)$ for all $g \in G$.

We claim that

$$S^G \subseteq Z \otimes_{\mathbb{k}} \mathbb{k}[G] . \tag{1}$$

To see this, let $s = \sum_1^r c_i \otimes f_i \in S^G$ with $\{f_i\}_1^r \subseteq \mathbb{k}[G]$ chosen \mathbb{k} -linearly independent. Choose $\{x_j\}_1^r \subseteq G$ such that the matrix $A_g = (f_i(x_jg))_{i,j}$ is invertible and put $B = (f_i(x_j))_{i,j}$. Then the equations $\sum_i c_i f_i(x) = \sum_i (g.c_i) f_i(xg)$ for all $x, g \in G$, with $x = x_j$ for $j = 1, \dots, r$, can be written as the matrix equation $(c_1, \dots, c_r)B = (g.c_1, \dots, g.c_r)A_g$ or else $(g.c_1, \dots, g.c_r) = (c_1, \dots, c_r)BA_g^{-1}$. This shows that $G.c_i \subseteq \sum_{i=1}^r \mathbb{k}c_i$ and hence $c_i \in Z$, proving (1).

Next, we show that there is an isomorphism

$$\varepsilon: S^G \xrightarrow{\sim} Z \quad (2)$$

such that

$$\varepsilon \circ (1_Z \otimes \rho_l(g)) = \rho(g) \circ \varepsilon \quad (3)$$

holds for all $g \in G$. Indeed, by [5, Lemma 18(b)], the G -action on Z is rational: it arises from a map of \mathbb{k} -algebras $\Delta_Z: Z \rightarrow Z \otimes_{\mathbb{k}} \mathbb{k}[G]$, $c \mapsto \sum c_0 \otimes c_1$, via $g.c = \sum c_0 c_1(g)$. By [6, equations (17) and (18)], the $\mathbb{k}[G]$ -linear extension of Δ_Z , which will also be denoted by Δ_Z , is an isomorphism of $\mathbb{k}[G]$ -algebras

$$\Delta_Z: Z \otimes_{\mathbb{k}} \mathbb{k}[G] \xrightarrow{\sim} Z \otimes_{\mathbb{k}} \mathbb{k}[G]$$

that satisfies the “intertwining formula” $\Delta_Z \circ (\rho \otimes \rho_r)(g) = (1_Z \otimes \rho_r)(g) \circ \Delta_Z$ for each $g \in G$. It follows that Δ_Z yields an isomorphism of $S^G = (Z \otimes_{\mathbb{k}} \mathbb{k}[G])^G$ with the sub algebra of $(1_Z \otimes \rho_r)(G)$ -invariants in $Z \otimes_{\mathbb{k}} \mathbb{k}[G]$. Since the latter algebra is clearly Z , the isomorphism (2) follows. It is easy to see that the isomorphism (2) is just the restriction to S^G of the Hopf counit $S \rightarrow \mathbb{C}$, $s \mapsto s(1)$. In particular, for any $s \in S^G$, we have

$$(\varepsilon \circ (1_Z \otimes \rho_l(g)))(s) = s(g^{-1}) = g.s(1) = (\rho(g) \circ \varepsilon)(s),$$

proving (3).

2.3. The case of an algebraic torus. Now let $G \cong (\mathbb{k}^\times)^d$ be an algebraic torus over \mathbb{k} and let $\Lambda = X(G) \cong \mathbb{Z}^d$ be its lattice of rational characters. Then $\mathbb{k}[G] = \mathbb{k}\Lambda$, the group algebra of Λ over \mathbb{k} . As it is customary to use additive notation for the lattice Λ , we will write the standard \mathbb{k} -basis of $\mathbb{k}[G]$ as $\{\mathbf{x}^\lambda \mid \lambda \in \Lambda\}$; so $\mathbf{x}^\lambda \mathbf{x}^{\lambda'} = \mathbf{x}^{\lambda+\lambda'}$ and $\mathbf{x}^\lambda(g) = \langle \lambda, g \rangle \in \mathbb{k}^\times$ for $g \in G$. Then $\rho_r(g)\mathbf{x}^\lambda = \langle \lambda, g \rangle \mathbf{x}^\lambda$ and

$$S = \bigoplus_{\lambda \in \Lambda} \mathcal{C} \otimes_{\mathbb{k}} \mathbb{k}\mathbf{x}^\lambda \cong \mathcal{C}\Lambda,$$

the group algebra of Λ over the field \mathcal{C} . Consider an element $s = \sum_{\lambda} s_{\lambda} \otimes \mathbf{x}^\lambda \in S$ with $s_{\lambda} \in \mathcal{C}$. Then $g.s = \sum_{\lambda} g.s_{\lambda} \otimes \langle \lambda, g \rangle \mathbf{x}^\lambda$ for $g \in G$. Hence, $s \in S^G$ if and only if $g.s_{\lambda} = \langle -\lambda, g \rangle s_{\lambda}$ for all g, λ . Putting $\mathcal{C}_{\lambda} = \{c \in \mathcal{C} \mid g.c = \langle \lambda, g \rangle c \text{ for all } g \in G\}$ and noting that each nonzero \mathcal{C}_{λ} is 1-dimensional over the fixed field \mathcal{C}^G , we have

$$S^G = \bigoplus_{\lambda \in \Lambda} \mathcal{C}_{-\lambda} \otimes_{\mathbb{k}} \mathbb{k}\mathbf{x}^\lambda \cong \mathcal{C}^G \Gamma, \quad (4)$$

the group algebra of the sublattice $\Gamma = \{\lambda \in \Lambda \mid \mathcal{C}_{\lambda} \neq 0\}$ over \mathcal{C}^G . We remark that $\Gamma = X(G/N)$ is the character lattice of the torus G/N , where N is the kernel of the action of G on Z ; so $Z = \bigoplus_{\lambda \in \Gamma} \mathcal{C}_{\lambda}$. The isomorphisms (4) and (2) prove part (a) of the Theorem. Note also the S is free over S^G .

We claim that each G -stable ideal \mathfrak{a} of S is generated by its intersection with S^G :

$$\mathfrak{a} = (\mathfrak{a} \cap S^G)S. \quad (5)$$

For the nontrivial inclusion \subseteq , let $s = \sum_{\lambda} s_{\lambda} \otimes \mathbf{x}^\lambda \in \mathfrak{a}$ be given. In order to show that $s \in (\mathfrak{a} \cap S^G)S$, we argue by induction on the size of $\text{Supp}(s) = \{\lambda \in \Lambda \mid s_{\lambda} \neq 0\}$, the length of s . Our claim being clear for $s = 0$, assume that $s \neq 0$. Suppose there exists an element $0 \neq t \in \mathfrak{a}$ with $\text{Supp}(t) \subsetneq \text{Supp}(s)$. Multiplying t and s with suitable units of the form $c \otimes \mathbf{x}^\mu$, we may assume that $0 \in \text{Supp}(t)$ and $t_0 = s_0 = 1$. Since t and $s - t$ are shorter than s , they both belong to $(\mathfrak{a} \cap S^G)S$ and hence $s \in (\mathfrak{a} \cap S^G)S$ as well. Therefore, we may assume that if $t \in \mathfrak{a}$ satisfies $\text{Supp}(t) \subsetneq \text{Supp}(s)$ then $t = 0$. Continuing to assume that $s_0 = 1$, this holds in particular for $t = s - g.s = \sum_{0 \neq \lambda} (s_{\lambda} - \langle \lambda, g \rangle s_{\lambda}) \otimes \mathbf{x}^\lambda$ for each $g \in G$. Therefore, we must have $s \in S^G$ and (5) is proved.

Now let \mathfrak{b} be an ideal of S^G and let \mathfrak{a} denote sum of all ideals of S that contract to \mathfrak{b} . Since S is free over S^G , we have $\mathfrak{a} \cap S^G = \mathfrak{b}$. Moreover, \mathfrak{a} is clearly G -stable and so (5) gives that $\mathfrak{a} = \mathfrak{b}S$. Thus, $\mathfrak{b}S$ is the unique largest ideal of S that contracts to \mathfrak{b} .

2.4. The prime correspondence. We start with some reminders from [6]. For now, let G again be an arbitrary connected affine algebraic group over \mathbb{k} and let $\mathbb{k}(G) = \text{Fract } \mathbb{k}[G]$ be the field of rational functions of G . The G -action on S via $\rho \otimes \rho_r$ extends uniquely to an action of G on the following localization of S :

$$T = \mathcal{C} \otimes_{\mathbb{k}} \mathbb{k}(G) .$$

Let $\text{Spec}^G(T)$ denote the collection of all G -stable prime ideals of T . Then [6, Theorem 9] establishes an order isomorphism

$$c: \text{Spec}_0 R \xrightarrow{\sim} \text{Spec}^G(T) \quad (6)$$

with the following G -equivariance property, for $P \in \text{Spec}_0 R$ and $g \in G$:

$$c(g.P) = (1_{\mathcal{C}} \otimes \rho_{\ell}(g))(c(P)) . \quad (7)$$

Since T is the localization S at the nonzero elements of $\mathbb{k}[G]$, contraction and extension yields a G -equivariant order isomorphism $\text{Spec } T \xrightarrow{\sim} \{\mathfrak{p} \in \text{Spec } S \mid \mathfrak{p} \cap \mathbb{k}[G] = 0\}$. Note that $\mathbb{k}[G]$ is a G -simple ring, because G acts transitively on itself by right multiplication. Therefore, each $\mathfrak{p} \in \text{Spec}^G(S)$ satisfies $\mathfrak{p} \cap \mathbb{k}[G] = 0$, and hence the above bijection restricts to a bijection

$$\text{Spec}^G(T) \xrightarrow{\sim} \text{Spec}^G(S) , \quad (8)$$

given by contraction and extension.

We now return to the case of an algebraic torus G . Then the map $\mathfrak{p} \mapsto \mathfrak{p} \cap S^G$ injects $\text{Spec}^G(S)$ into $\text{Spec}(S^G)$ by (5). Moreover, for any $\mathfrak{q} \in \text{Spec}(S^G)$, we know that $\mathfrak{p} = \mathfrak{q}S$ is the unique largest ideal of S that contracts to \mathfrak{q} , which implies that $\mathfrak{p} \in \text{Spec}^G(S)$. Thus we obtain a bijection

$$\text{Spec}^G(S) \xrightarrow{\sim} \text{Spec}(S^G) \quad (9)$$

that is again given by contraction and extension. From (6) - (9) in conjunction with the isomorphism (2) we obtain the desired order isomorphism $\gamma: \text{Spec}_0 R \xrightarrow{\sim} \text{Spec}(Z)$. The fact that γ is G -equivariant is immediate from the G -equivariance property (7) of the bijection (6) and the intertwining formula (3). This completes the proof of the Stratification Theorem.

REFERENCES

1. Kenneth A. Brown and Kenneth R. Goodearl, *Lectures on algebraic quantum groups*, Advanced Courses in Mathematics. CRM Barcelona, Birkhäuser Verlag, Basel, 2002. MR 1898492 (2003f:16067)
2. K. R. Goodearl, *Prime spectra of quantized coordinate rings*, Interactions between ring theory and representations of algebras (Murcia), Lecture Notes in Pure and Appl. Math., vol. 210, Dekker, New York, 2000, pp. 205–237. MR 1759846 (2001b:16052)
3. K. R. Goodearl and E. S. Letzter, *The Dixmier-Moeglin equivalence in quantum coordinate rings and quantized Weyl algebras*, Trans. Amer. Math. Soc. **352** (2000), no. 3, 1381–1403. MR 1615971 (2000j:16040)
4. K. R. Goodearl and J. T. Stafford, *The graded version of Goldie's theorem*, Algebra and its applications (Athens, OH, 1999), Contemp. Math., vol. 259, Amer. Math. Soc., Providence, RI, 2000, pp. 237–240. MR 1780524 (2001g:16087)
5. Martin Lorenz, *Group actions and rational ideals*, Algebra Number Theory **2** (2008), no. 4, 467–499. MR 2411408 (2009i:16058)
6. ———, *Algebraic group actions on noncommutative spectra*, Transform. Groups **14** (2009), no. 3, 649–675. MR 2534802 (2010m:14060)

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